

HYPERBOLICITY CONES OF ELEMENTARY SYMMETRIC POLYNOMIALS ARE SPECTRAHEDRAL

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ABSTRACT. We prove that the hyperbolicity cones of elementary symmetric polynomials are spectrahedral, i.e., they are slices of the cone of positive semidefinite matrices. The proof uses the matrix-tree theorem, an idea already present in Choe *et al.* [2].

1. INTRODUCTION AND MAIN RESULTS

A homogenous polynomial $h(\mathbf{x}) \in \mathbb{R}[x_1, \dots, x_n]$ is *hyperbolic* with respect to a vector $\mathbf{e} \in \mathbb{R}^n$ if $h(\mathbf{e}) \neq 0$ and for all $\mathbf{x} \in \mathbb{R}^n$, the univariate polynomial $t \mapsto h(\mathbf{x} + t\mathbf{e})$ has only real zeros. The *hyperbolicity cone*, $\Lambda_+(h, \mathbf{e})$, is the closure of the connected component of $\{\mathbf{x} \in \mathbb{R}^n : h(\mathbf{x}) \neq 0\}$ which contains \mathbf{e} . The notion of hyperbolic polynomials originates from PDE-theory and the work of Petrovsky and Gårding. However, during the last fifteen years there has been increasing interest in hyperbolic polynomials from unexpected areas such as combinatorics and convex optimization [2, 4, 8]. Optimization over hyperbolicity cones was first considered by Güler [4] and a rich theory for hyperbolic programs has been developed [4, 8, 9] which extends many features of semidefinite programming.

An important open question regarding hyperbolic programming concerns the generality of hyperbolicity cones. The most fundamental example of a hyperbolic polynomial is the determinant $h(\mathbf{x}) = \det(X)$, where $X = (x_{ij})_{i,j=1}^m$ is the symmetric $m \times m$ matrix of $m(m+1)/2$ variables and $\mathbf{e} = I$ is the identity matrix. Hence the cone of positive semidefinite $m \times m$ matrices is a hyperbolicity cone, and it follows that so are *spectrahedral cones*, i.e., cones of the form

$$\left\{ \mathbf{x} \in \mathbb{R}^n : \sum_{i=1}^n x_i A_i \text{ is positive semidefinite} \right\}, \quad (1.1)$$

where A_i , $1 \leq i \leq n$, are symmetric $m \times m$ matrices. It has been speculated in whether the converse is true [5, 8]:

Conjecture 1.1 (Generalized Lax conjecture). *All hyperbolicity cones are spectrahedral i.e., of the form (1.1).*

The evidence in favor of Conjecture 1.1 are not overwhelming:

- (1) It is true for hyperbolic polynomials in three variables [5, 6],

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(2) It is true for quadratic polynomials [7].

Stronger conjectures that imply Conjecture 1.1 were recently disproved in [1], see also [7].

In this note we are concerned with the hyperbolicity cones of the *elementary symmetric polynomials*:

$$e_k(x_1, \dots, x_n) = \sum_{1 \leq i_1 < \dots < i_k \leq n} x_{i_1} \cdots x_{i_k}.$$

They are hyperbolic with respect to $\mathbf{1} = (1, \dots, 1)^T$, and their hyperbolicity cones contain the positive orthant. Zinchenko [12] studied the hyperbolicity cones of elementary symmetric polynomials and proved that they are *spectrahedral shadows*, i.e., projections of spectrahedral cones. Sanyal [10] proved that the hyperbolicity cone of $e_{n-1}(x_1, \dots, x_n)$ is spectrahedral and conjectured that all hyperbolicity cones of elementary symmetric polynomials are spectrahedral, although it is subsumed by Conjecture 1.1. We will prove this conjecture:

Theorem 1.2. *Hyperbolicity cones of elementary symmetric polynomials are spectrahedral.*

The hyperbolicity cone of $h(\mathbf{x})$ is spectrahedral if and only if there is a pencil $\sum_{i=1}^n x_i A_i$ of symmetric matrices and a homogeneous polynomial $q(\mathbf{x})$ such that

$$q(\mathbf{x})h(\mathbf{x}) = \det \left(\sum_{i=1}^n x_i A_i \right), \quad (1.2)$$

and $\Lambda_+(q, \mathbf{e}) \supseteq \Lambda_+(h, \mathbf{e})$. Our idea to prove Theorem 1.2 was to use the matrix-tree theorem (Theorem 2.1 below) to construct a graph for which the spanning tree polynomial is a multiple of the elementary symmetric polynomial in question. In the process we became aware of that the same idea was already present in [2, Section 9.1] where it was observed that the elementary symmetric polynomials are factors of determinantal polynomials. However, to prove Theorem 1.2 we need to know the other factors, and that $\Lambda_+(q, \mathbf{e}) \supseteq \Lambda_+(h, \mathbf{e})$ holds.

Recall that a cone is *polyhedral* if it is the intersection of a finite number of half-spaces, i.e., it is the hyperbolicity cone of a polynomial of the form $h(\mathbf{x}) = \ell_1(\mathbf{x}) \cdots \ell_d(\mathbf{x})$ where $\ell_j(\mathbf{x})$ is a linear form for $1 \leq j \leq d$. If h is hyperbolic with respect to $\mathbf{e} = (e_1, \dots, e_n)^T$, then

$$D_{\mathbf{e}}h(\mathbf{x}) = \sum_{i=1}^n e_i \frac{\partial}{\partial x_i} h(\mathbf{x})$$

is hyperbolic with respect to \mathbf{e} and $\Lambda_+(D_{\mathbf{e}}h, \mathbf{e}) \supseteq \Lambda_+(h, \mathbf{e})$, see e.g. [3, 8]. Of course polyhedral cones are spectrahedral and it is natural to ask if the derivative cones $\Lambda_+(D_{\mathbf{e}}^k h, \mathbf{e})$ are also spectrahedral for $1 \leq k \leq d-1$. For $k=1$ this was answered to the affirmative by Sanyal [10]. Using Theorem 1.2 we settle the remaining cases:

Corollary 1.3. *The derivative cones of polyhedral cones are spectrahedral, i.e., if $h(\mathbf{x}) = \ell_1(\mathbf{x}) \cdots \ell_d(\mathbf{x})$ and $h(\mathbf{e}) \neq 0$, then $\Lambda_+(D_{\mathbf{e}}^k h, \mathbf{e})$ is spectrahedral for each $1 \leq k \leq d-1$.*

Proof. Since

$$D_{\mathbf{e}}^k h(\mathbf{x}) = k! h(\mathbf{e}) e_{d-k}(\ell_1(\mathbf{x}), \dots, \ell_d(\mathbf{x})),$$

see e.g. [8, Proposition 18], the corollary follows immediately from Theorem 1.2. \square

2. PROOF OF THEOREM 1.2

Let $G = (V, E)$ be a finite connected graph with multiple edges allowed, and assign variables $\mathbf{x} = \{x_e\}_{e \in E}$ to the edges. Recall that a *spanning tree* is a maximal, with respect to inclusion, subset T of E that contains no cycle. The *spanning tree polynomial* is defined as

$$T_G(\mathbf{x}) = \sum_T \prod_{j \in T} x_j,$$

where the sum is over all spanning trees in G . Suppose $V = [n] := \{1, \dots, n\}$ and let $\{\delta_i\}_{i=1}^n$ be the standard bases of \mathbb{R}^n . The *weighted Laplacian* of G is defined as

$$L_G(\mathbf{x}) = \sum_{e \in E} x_e (\delta_{e_1} - \delta_{e_2})(\delta_{e_1} - \delta_{e_2})^T,$$

where e_1 and e_2 are the vertices incident to $e \in E$. We refer to [11, Theorem VI.29] for a proof of the next classical theorem that goes back to Kirchhoff and Maxwell.

Theorem 2.1 (Matrix–tree theorem). *For $i \in V$, let $L_G(\mathbf{x})_{ii}$ be the matrix obtained by deleting the column and row indexed by i in $L_G(\mathbf{x})$. Then*

$$T_G(\mathbf{x}) = \det(L_G(\mathbf{x})_{ii}).$$

Remark 2.2. Let $G = K_{n+1}$, the complete graph on $n+1$ vertices. Then

$$L_G(\mathbf{x})_{(n+1)(n+1)} = (v_{ij})_{i,j=1}^n,$$

where $v_{ij} = -x_{ij}$ if $i \neq j$ and $v_{ii} = x_{i(n+1)} - x_{ii} + \sum_{j=1}^n x_{ij}$. Hence the hyperbolicity cone of $T_G(\mathbf{x})$ is linearly isomorphic to the cone of positive semidefinite $n \times n$ matrices. Thus the generalized Lax conjecture is equivalent to that each hyperbolicity cone is a slice of a hyperbolicity cone of some spanning tree polynomial. This is the reason for why we believed that at least for elementary symmetric polynomials one would be able to use the matrix–tree theorem to deduce that the hyperbolicity cones are spectrahedral.

Let $\{x_j\}_{j=1}^\infty$ be independent variables, and for a finite non-empty set $S \subset \mathbb{Z}_+ := \{1, 2, \dots\}$ let

$$e_k(S) = \sum_{\substack{T \subseteq S \\ |T|=k}} \prod_{j \in T} x_j,$$

be the k th elementary symmetric polynomial in $\{x_j\}_{j \in S}$. For $k \geq 1$ let $q_k(S) = e_k(S)/e_{k-1}(S)$. The recursion

$$kq_k(S) = \sum_{j \in S} \frac{x_j q_{k-1}(S \setminus \{j\})}{x_j + q_{k-1}(S \setminus \{j\})}, \quad \text{for all } k \geq 2, \quad (2.1)$$

follows from *Euler's formula* for homogenous polynomials of degree k :

$$kh(\mathbf{x}) = \sum_{j=1}^n x_j \frac{\partial h}{\partial x_j},$$

and the recursion

$$e_k(S) = e_k(S \setminus \{j\}) + x_j e_{k-1}(S \setminus \{j\}).$$

It is no accident that (2.1) is reminiscent of the operation (C) on spanning tree polynomials below:

- (A) If we replace an edge $e \in E$ between vertices i and j with k parallel edges e_1, \dots, e_k between i and j the resulting polynomial is obtained by setting $x_e = x_{e_1} + \dots + x_{e_k}$ in T_G .
- (B) If an edge e between i and j is replaced by a path i, e_1, k, e_2, j then the resulting polynomial is

$$(x_{e_1} + x_{e_2})T_G(\mathbf{x}) \Big|_{x_e = x_{e_1}x_{e_2}/(x_{e_1} + x_{e_2})}.$$

- (C) By (A) and (B), if we replace an edge e between i and j by a series parallel graph as in Fig. 1 with edge-variables $x_1, y_1, \dots, x_m, y_m$, then the resulting polynomial is obtained by multiplying by $\prod_{j=1}^m (x_j + y_j)$ and setting

$$x_e = \sum_{j=1}^m \frac{x_j y_j}{x_j + y_j},$$

in T_G .

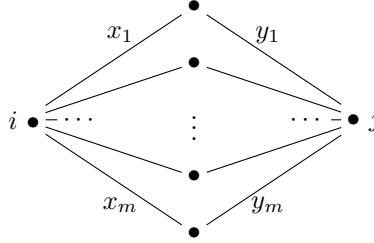


FIGURE 1.

Let $G_{n,0} = s \bullet - \bullet z$, and for $n \geq k \in \mathbb{Z}_+$ let $G_{n,k}$ be the graph with vertices consisting of two designated vertices s and z , and all words $w = w_1 w_2 \dots w_\ell$ such that $1 \leq \ell \leq k$, $w_i \in [n]$ for all i and $w_i \neq w_j$ for all $1 \leq i < j \leq \ell$. The edges in $G_{n,k}$ are between the vertices:

- (1) s and i for all $1 \leq i \leq n$;
- (2) $w_1 \dots w_{i-1}$ and $w_1 \dots w_{i-1} w_i$ for all $1 \leq i \leq k$;
- (3) $w_1 \dots w_k$ and z ,

see Fig. 2. Let n be fixed and let $H_{k,r}(\mathbf{x}, \mathbf{y})$, $k \leq r$, be obtained from $T_{G_{n,k}}$ by setting the edge-variables as:

- $r!x_i$ if as in (1);
- $(r - i + 1)!x_{w_i}$ if as in (2);
- $y_{w_1 \dots w_k}$ if as in (3).

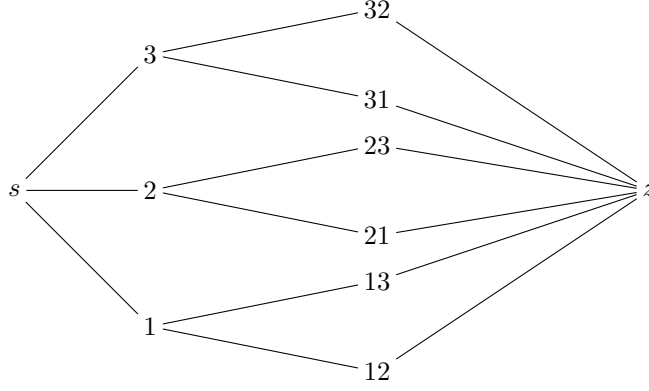
Let $W_{n,k}$ be the set of all words $w = w_1 w_2 \dots w_k$ such that $w_i \in [n]$ for all i and $w_i \neq w_j$ for all $1 \leq i < j \leq k$, and let

$$\mathbf{q}_{kr} = \{(r - k + 1)!q_{r-k+1}(\{w_1, \dots, w_k\}^c)\}_{w \in W_{n,k}},$$

where $S' = [n] \setminus S$. Let further

$$\gamma_{kr} = \prod_{I \in \binom{[n]}{n-k}} e_{r-k}(I)^{k!},$$

where $\binom{[n]}{j} = \{S \subseteq [n] : |S| = j\}$. Note that $\gamma_{0r} = e_r(x_1, \dots, x_n)$.

FIGURE 2. The graph $G_{3,2}$.

Lemma 2.3. *Let $1 \leq k \leq r \leq n-1$ be integers. Then there are constants $C_{i,r}$, $0 \leq i \leq r$ such that*

$$H_{0,r}(\mathbf{x}, \mathbf{q}_{0r}) = C_{0,r} \frac{e_{r+1}(\mathbf{x})}{e_r(\mathbf{x})},$$

and

$$H_{k,r}(\mathbf{x}, \mathbf{q}_{kr}) = C_{k,r} H_{k-1,r}(\mathbf{x}, \mathbf{q}_{(k-1)r}) \frac{(\gamma_{(k-1)r})^{n-k+1}}{\gamma_{kr}},$$

for $k > 0$.

Proof. The first statement follows immediately from (2.1) and (C). Using (2.1) and (C) for each rightmost subgraph such as in Fig. 1 we get

$$H_{k,r}(\mathbf{x}, \mathbf{q}_{kr}) = Q_{k,r}(\mathbf{x}) H_{k-1,r}(\mathbf{x}, \mathbf{q}_{(k-1)r}),$$

where $Q_{k,r}$ is the product of all the factor that comes from the operation in (C). Hence, modulo constants, for each injective word $w_1 \cdots w_k \in W_{n,k}$ we get a factor

$$x_{w_k} + q_{r-k+1}(\{w_1, \dots, w_k\}') = \frac{e_{r-k+1}(\{w_1, \dots, w_{k-1}\}')}{e_{r-k}(\{w_1, \dots, w_k\}')}.$$

Thus collecting each possible nominator and denominator we see that $Q_{k,r} = (\gamma_{(k-1)r})^{n-k+1} / \gamma_{kr}$. □

Lemma 2.4. *Let $1 \leq k \leq n-1$. Then*

$$H_{k,k}(\mathbf{x}, \mathbf{q}_{kk}) = C_k e_{k+1}(\mathbf{x}) \prod_{\substack{S \subseteq [n] \\ |S| \leq k-1}} (\partial^S e_k(\mathbf{x}))^{|S|!(n-|S|-1)}, \quad (2.2)$$

where C_k is a constant and $\partial^S = \prod_{i \in S} \partial / \partial x_i$.

Proof. By iterating Lemma 2.3

$$H_{k,k}(\mathbf{x}, \mathbf{q}_{kk}) = C_k e_{k+1}(\mathbf{x}) \prod_{j=0}^{k-1} (\gamma_{jk})^{n-j-1}.$$

The theorem follows by noting that $\partial^S e_k(\mathbf{x}) = e_{k-|S|}(S')$. □

Let us state some fundamental properties of hyperbolicity cones for reference. For proofs we refer to [3, 8]. Let $\Lambda_{++}(h, \mathbf{e})$ be the interior of $\Lambda_+(h, \mathbf{e})$.

Proposition 2.5. *Let h be hyperbolic with respect to \mathbf{e} .*

- (1) $\mathbf{x} \in \Lambda_{++}(h, \mathbf{e})$ if and only if all zeros of the univariate polynomial $t \mapsto h(\mathbf{x} + t\mathbf{e})$ are negative;
- (2) If $\mathbf{v} \in \Lambda_{++}(h, \mathbf{e})$, then h is hyperbolic with respect to \mathbf{v} and $\Lambda_+(h, \mathbf{v}) = \Lambda_+(h, \mathbf{e})$;
- (3) h is hyperbolic to $-\mathbf{e}$ and $\Lambda_+(h, -\mathbf{e}) = -\Lambda_+(h, \mathbf{e})$.

Lemma 2.6. *Suppose that h is hyperbolic with respect to \mathbf{e} and $\mathbf{v} \in \Lambda_+(h, \mathbf{e})$ is such that $D_{\mathbf{v}}h \neq 0$. Then $D_{\mathbf{v}}h$ is hyperbolic with respect to \mathbf{e} and $\Lambda_+(h, \mathbf{e}) \subseteq \Lambda_+(D_{\mathbf{v}}h, \mathbf{e})$.*

Proof. If $\mathbf{v} \in \Lambda_{++}(h, \mathbf{e})$, then the conclusion is known to follow, see e.g. [3, Theorem 4]. Let \mathbf{v} be on the boundary of $\Lambda_+(h, \mathbf{e})$ and $\mathbf{e}' \in \Lambda_{++}(h, \mathbf{e})$. Then h is hyperbolic with respect to \mathbf{e}' , by Proposition 2.5 (2), so we may in fact assume that $\mathbf{e} = \mathbf{e}'$. Hence, for all $\mathbf{x} \in \mathbb{R}^n$, all zeros of $t \mapsto D_{\mathbf{v}}h(\mathbf{x} + t\mathbf{e})$ are real (unless $D_{\mathbf{v}}h(\mathbf{x} + t\mathbf{e}) \equiv 0$ in t) by Hurwitz' theorem on the continuity of zeros. It remains to prove that $D_{\mathbf{v}}h(\mathbf{e}) \neq 0$. Consider the univariate polynomial

$$p(t) = h(\mathbf{e} + t\mathbf{v}) = h(\mathbf{e}) + tD_{\mathbf{v}}h(\mathbf{e}) + \dots$$

Since $\mathbf{v} \in \Lambda_+(h, \mathbf{e})$ all zeros of $p(t)$ are real and non-positive. Hence if $D_{\mathbf{v}}h(\mathbf{e}) = 0$, then $h(\mathbf{e} + t\mathbf{v}) = p(t) \equiv h(\mathbf{e})$, and thus $h(\mathbf{v} + t\mathbf{e}) = h(\mathbf{e})t^d$ where d is the degree of h . By Proposition 2.5 this holds if and only if $\mathbf{v} \in \Lambda_+(h, \mathbf{e}) \cap (-\Lambda_+(h, \mathbf{e}))$, and thus $h(\mathbf{v} + t\mathbf{y}) = h(\mathbf{y})t^d$ for all $\mathbf{y} \in \Lambda_{++}(h, \mathbf{e})$ by Proposition 2.5 (2). Since $\Lambda_{++}(h, \mathbf{e})$ is open $h(\mathbf{x} + t\mathbf{v}) \equiv h(\mathbf{x})$ for all $\mathbf{x} \in \mathbb{R}^n$, and in particular $D_{\mathbf{v}}h(\mathbf{x}) \equiv 0$ which contradicts the assumptions. \square

Proof of Theorem 1.2. Note that

$$\mathbf{q}_{kk} = \left\{ \sum_{j \notin \{w_1, \dots, w_k\}} x_j \right\}_{w \in W_{n,k}},$$

so that (2.2) in Lemma 2.4 is a determinantal polynomial by the matrix-tree theorem. It remains to prove $\Lambda_+(e_{k+1}) \subseteq \Lambda_+(\partial^S e_k)$. Now $e_k(\mathbf{x}) = (n-k)^{-1}D_{\mathbf{1}}e_{k+1}(\mathbf{x})$, so $\Lambda_+(e_{k+1}) \subseteq \Lambda_+(e_k)$ by Lemma 2.6. Since the coordinate directions are in $\Lambda_+(e_k)$, we have $\Lambda_+(e_k) \subseteq \Lambda_+(\partial^S e_k)$ by Lemma 2.6. \square

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